
Student Number

2023 Glenwood High School
Year 12 – Trial HSC Examination
Assessment Task 4

Mathematics Extension 2

General Instructions

* Reading Time – 10 minutes
* Working time – 3 hours
* Write using black pen
* NESA approved calculators may be used
* A reference sheet is provided
* For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3 – 7)

* Attempt Questions 1-10
* Allow about 15 minutes for this section

Section II – 90 marks (pages 8 – 15)

* Attempt Questions 11-16
* Allow about 2 hours and 45 minutes for this section

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Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. Which of the following is a solution to the equation $\left|e^{-i\theta}-1\right|=2$?

A. $\theta=-\frac{\pi}{2}$

B. $\theta=0$

C. $\theta=\frac{\pi}{2}$

D. $\theta=\pi$

2. What is the contrapositive of the following statement?

"If you're happy and you know it, then you will clap your hands."

A. If you clap your hands, then you're happy and you know it.

B. If you clap your hands, then you're either not happy or you don't know it.

C. If you don't clap your hands, then you're not happy and you don't know it.

D. If you don't clap your hands, then you're either not happy or you don't know it.

3. It is given that a, b, c and d are consecutive integers.

Which of the following statements may be false?

A. $abcd$ is divisible by 3

B. $abcd$ is divisible by 8

C. $a+b+c+d$ is divisible by 2

D. $a+b+c+d$ is divisible by 4

4. Vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ have components $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ t \\ 4 \end{pmatrix}$ respectively.

The following statements are made about $\underset{\sim}{u}$ and $\underset{\sim}{v}$:

Statement I: When $t = 5$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel

Statement II: When $t = \frac{1}{5}$, $\underset{\sim}{v}$ is a unit vector

Which of the following is true?

A. Both statements are incorrect.

B. Only Statement I is correct.

C. Only Statement II is correct.

D. Both statements are correct.

5. $\int \tan^3 x \, dx$ can be evaluated as:

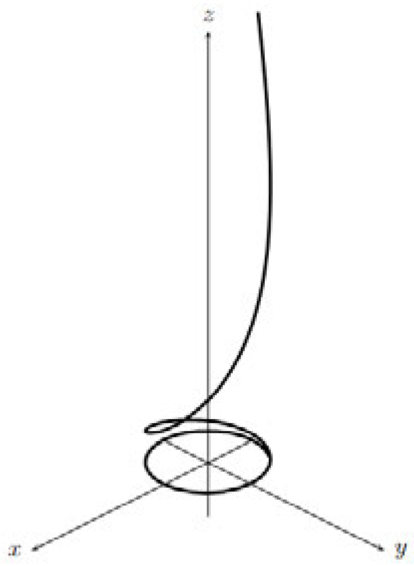
A. $\frac{1}{2}\tan^2 x - \ln|\cos x| + C$

B. $\frac{1}{2}\tan^2 x + \ln|\cos x| + C$

C. $\frac{1}{2}\tan^2 x + C$

D. $\frac{1}{4}\tan^4 x + C$

6. Which vector equation best describes the curve below?



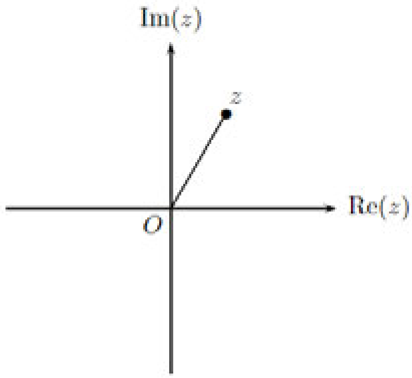
A. $\underset{\sim}{r}(t) = \sin t\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j} + t^2\underset{\sim}{k}$

B. $\underset{\sim}{r}(t) = \sin t\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j} + \ln t\,\underset{\sim}{k}$

C. $\underset{\sim}{r}(t) = \cos t\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j} + t\underset{\sim}{k}$

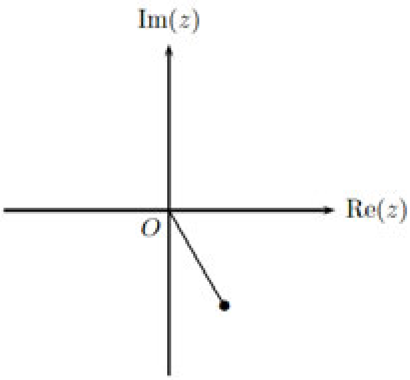
D. $\underset{\sim}{r}(t) = \cos t\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j} + 2^t\underset{\sim}{k}$

7. The complex number z is shown below.

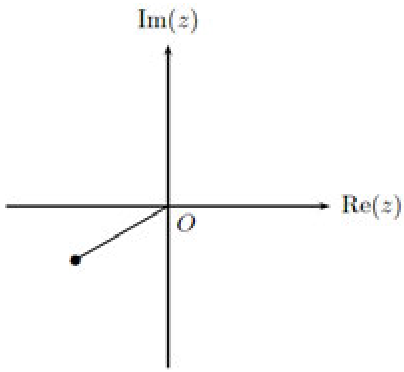


Which of the following is the graph of $i\bar{z}$?

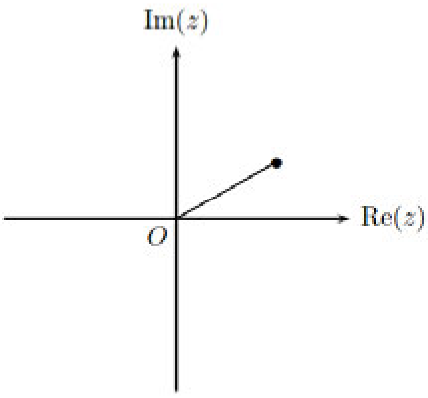
A.



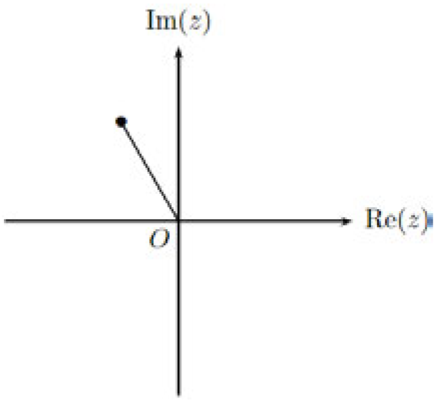
B.



C.



D.



8. The velocity v cm/s of a particle moving along the x −axis is given by

$$v^2 = -4x^2 + 24x - 34$$

where x is in centimetres.

Given the particle is moving in simple harmonic motion, find the centre of the motion.

A. -34

B. -3

C. 3

D. 4

9. The diameter of a sphere is a line segment joining $(2, -6, 1)$ and $(-6, a, 9)$.

Given the volume of the sphere is 288π cubic units, find a possible value for a.

A. -2

B. 0

C. 2

D. 6

10. $f(x)$ and $g(x)$ are continuous functions such that:

$$f(x) = f(1-x)$$

$$g(x) + g(1-x) = 2$$

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$$

Which one of the following is equivalent to:

$$\int_0^1 f(x)\,g(x)dx$$

A. $\int_0^1 f(x)\,dx$

B. $2\int_0^1 f(x)\,dx$

C. $3\int_0^1 f(x)\,dx$

D. $4\int_0^1 f(x)\,dx$

Section II

90 marks

Attempt Questions 11 – 16.

Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a new writing booklet.

(a) (i) Express $z = \frac{1+\sqrt{3}i}{1+i}$ in the form $r(\cos\theta + i\sin\theta)$. **2**

(ii) Find the smallest positive integer n such that z^n is a real number. **2**

(b) Find the values of a and b such that $(3, a, 5)$ divides the line segment joining $(-1, 0,3)$ and $(5, 3, b)$ in the ratio $2{:}\,1$. **3**

(c) (i) Find the square roots of $-3 - 4i$. **2**

(ii) Hence, or otherwise, solve the equation $z^2 - 3z + (3 + i) = 0$. **2**

(d) A particle travels along the x −axis so that the relationship between its velocity v cm/s and displacement x cm is given by $v = -0.3x$.

(i) Show that the particle does not exhibit simple harmonic motion. **1**

(ii) The initial displacement of the particle is $x = 6$ cm. **2**

How long does it take for the particle to have a displacement of 2 cm?

End of Question 11

Question 12 (15 marks) Use a new writing booklet.

(a) (i) Write the following mathematical statement in words. **1**

$$\forall\, k \in \mathbb{Z}^+, \exists\, x \in \mathbb{Z} \text{ such that } x^2 + x - k = 0$$

(ii) Use proof by contradiction to prove the following statement: **3**

"If p is odd, then $x^2 + x - p^2 = 0$ has no integer solution."

(b) Two lines, l_1 and l_2, are given below:

$$l_1: \underset{\sim}{r}(t) = 2\underset{\sim}{i} + 3\underset{\sim}{k} + t(\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$$

$$l_2: \frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$$

(i) Show that l_1 and l_2 intersect. **2**

(ii) Find the acute angle between l_1 and l_2, to the nearest minute. **2**

(c) A particle is moving in simple harmonic motion in a straight line with amplitude 8 metres. The speed of the particle is 12 m/s when it is 4 metres from the centre of its motion. **3**

Find the period of the motion.

Question 12 continues on page 10

(d) (i) Determine the values a, b and c such that **2**

$$\frac{1}{x(1+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{1+x^2}$$

(ii) Hence, find **2**

$$\int \frac{\arctan x}{x^2} dx$$

End of Question 12

Question 13 (16 marks) Use a new writing booklet.

(a) A particle is projected along the x −axis with speed u and has acceleration given by $a = -kv^3, k > 0$.

(i) Show that the particle's velocity is given by **3**

$$v = \frac{u}{1 + kux}$$

where x is the displacement from the starting point.

(ii) Find how long it takes for the particle to slow down to a speed of $\frac{u}{3}$. **2**

Write your answer in terms of k and u.

(b) Using an appropriate substitution, evaluate **3**

$$\int \frac{3}{x^2\sqrt{x^2 - 9}}\,dx$$

(c) The complex number z is represented by the point P. **4**

Given that P moves in a way such that $\frac{z-2}{z-i}$ is purely imaginary, sketch the locus of P, show any intercepts and any key features.

(d) Let ω be a non-real fifth root of unity.

(i) Show that $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$. **1**

(ii) Prove that $\omega - \omega^4$ is a root of the equation $z^4 + 5z^2 + 5 = 0$. **3**

End of Question 13

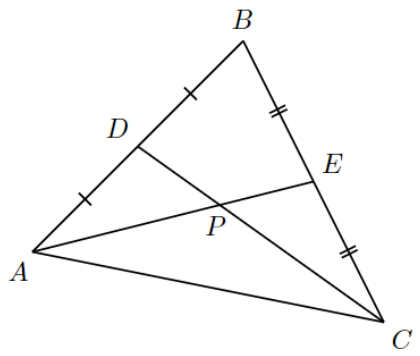
Question 14 (16 marks) Use a new writing booklet.

(a) Let a_n be the sequence defined recursively by $a_0 = 0$ and $a_n = a_{n-1} + 3n^2$ for all integers $n \geq 1$. **3**

Use mathematical induction to prove that for all integers $n \geq 0$,

$$a_n = \frac{n(n+1)(2n+1)}{2}$$

(b) The diagram below shows ΔABC. Points D and E bisects AB and BC respectively. Let $\overrightarrow{DB} = \underset{\sim}{u}$ and $\overrightarrow{BE} = \underset{\sim}{v}$.



(i) The lines AE and CD intersect at P such that $\overrightarrow{AP} = k\overrightarrow{AE}$, $0 < k < 1$. **4**

Using vector methods, show that $k = \frac{2}{3}$.

(ii) Let F be the point that bisects AC. **2**

Show that B, F and P are collinear.

Question 14 continues on page 13

(c) Given $z = \cos\theta + i\sin\theta$:

(i) Prove $z^n + \frac{1}{z^n} = 2\cos n\theta$. **2**

(ii) Hence, by considering the expansion of $\left(z + \frac{1}{z}\right)^4$, show that **3**

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

(iii) Hence, evaluate **2**

$$\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$$

End of Question 14

Question 15 (15 marks) Use a new writing booklet.

(a) Find the exact area bounded by the curve $y = 2x \ln x$, the x −axis and the lines $x = \frac{1}{e}$ and $x = e$. **4**

(b) (i) Prove by mathematical induction, for $z \neq 1$, **4**

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2}$$

(ii) Hence, or otherwise, prove that **3**

$$2 + \frac{3}{2} + \frac{4}{2^2} + \cdots + \frac{n}{2^{n-2}} = 6 - \frac{n+2}{2^{n-2}}$$

(iii) Show that, when $z \neq 0$, **1**

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{z^{-1} - (n+1)z^{n-1} + nz^n}{z^{-1} - 2 + z}$$

(iv) Hence, by writing $z = \cos\theta + i\sin\theta$ and using De Moivre's Theorem, show that **3**

$$\sum_{k=1}^{n} k\cos(k-1)\theta = \frac{(n+1)\cos(n-1)\theta - n\cos n\theta - \cos\theta}{2(1-\cos\theta)}$$

End of Question 15

Question 16 (14 marks) Use a new writing booklet.

(a) (i) Given that $a > 0, b > 0$, prove that **2**

$$a + b \geq 2\sqrt{ab}$$

(ii) Hence, show that $\sec^2 x \geq 2\tan x$. **1**

(iii) Prove that, for all integers $n \geq 0$ and $x \in \left(0, \frac{\pi}{2}\right)$, **3**

$$\sec^{2n} x + \operatorname{cosec}^{2n} x \geq 2^{n+1}$$

(b) Let $I_n = \int_0^a x^n \sqrt{a^2 - x^2}\, dx$, $a \in \mathbb{R}^+$ and $n = 0, 1, 2, \ldots$

(i) Prove that, for $n = 2, 3, 4, \ldots$ **3**

$$I_n = \frac{a^2(n-1)}{n+2} I_{n-2}$$

(ii) Prove that, for $n = 0, 1, 2, \ldots$ **3**

$$I_{2n} = \pi \left(\frac{a}{2}\right)^{2n+2} \frac{(2n)!}{n!\,(n+1)!}$$

(iii) $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the Catalan numbers. **2**

Prove that

$$C_n = \frac{1}{\pi}\int_0^2 x^{2n}\sqrt{4 - x^2}\, dx$$

End of Paper

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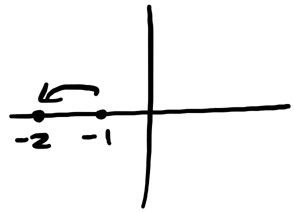
2023 Trial Examination 2023 Mathematics Extension 2 Solutions

1)



$\theta = \pi$

D

2) Contrapositive: $\sim Q \Rightarrow \sim P$

D

3) $k + (k+1) + (k+2) + (k+3) = 4k + 6$

$= 4(k+1) + 2$ which is not divisible by 4 D

4) $\frac{3}{1} \neq \frac{5}{3} \neq \frac{1}{2}$ $\therefore$ statement I is false

$\sqrt{3^2 + (\frac{1}{5})^2 + 4^2} = \sqrt{\frac{626}{25}} \neq 1$ $\therefore$ statement II is false

A

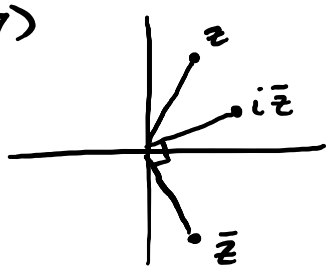
5) $\int \tan x \tan^2 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$

$= \int \tan x \sec^2 x - \tan x \, dx$

$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$ B

6) D

7)



C

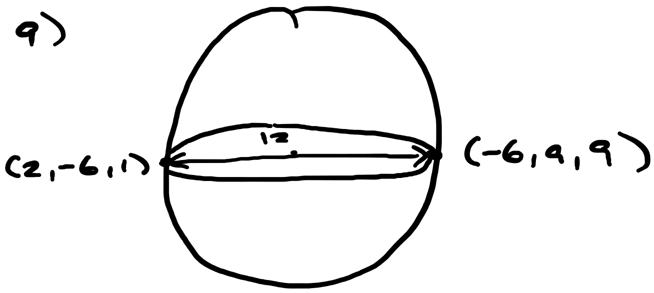
8) $\ddot{x} = \frac{d}{dx}(-2x^2 + 12x - 17)$

$= -4x + 12$

$= -4(x-3)$

$\therefore 3$ C

9)



$V = \frac{4}{3}\pi r^3 = 288\pi$

$r^3 = 216$

$r = 6$

$\sqrt{(-6-2)^2+(a+6)^2+(9-1)^2} = 12$

$(a+6)^2 + 128 = 144$

$(a+6)^2 = 16$

$a+6 = \pm 4$

$a = -2$ or -10 Ⓐ

10) $I = \int_0^1 f(1-x)\,[2-g(1-x)]\,dx$

$= \int_0^1 2f(1-x) - f(1-x)g(1-x)\,dx$

$= \int_0^1 2f(x)\,dx - \int_0^1 f(1-x)g(1-x)\,dx$

Let $u = 1-x$

x	u
0	1
1	0

$\frac{du}{dx} = -1$

$dx = -du$

$I = 2\int_0^1 f(x)\,dx - \int_1^0 f(u)g(u)(-du)$

$= 2\int_0^1 f(x)\,dx - \int_0^1 f(x)g(x)\,dx$

$2I = 2\int_0^1 f(x)\,dx$

$I = \int_0^1 f(x)\,dx$ Ⓐ

11a i) $1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\therefore \dfrac{1+\sqrt{3}i}{1+i} = \dfrac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}$

$= \frac{2}{\sqrt{2}}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

Many students wrote z in cartesian form to get $z = \frac{1+\sqrt{3}}{2} + \frac{-1+\sqrt{3}}{2}i$

ii) $z^n = \sqrt{2}^n\left(\cos\frac{n\pi}{12} + i\sin\frac{n\pi}{12}\right)$

Real $\Rightarrow \sin\frac{n\pi}{12} = 0$

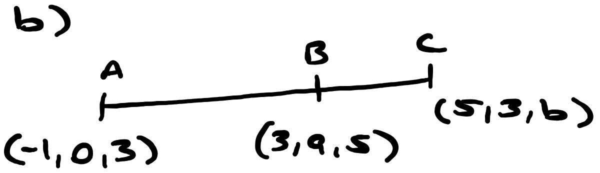
$\frac{n\pi}{12} = 0, \pi, 2\pi, \ldots$

$n = 0, 12, 24, \ldots$

$\therefore n = 12$

Some students wrote $\sin\frac{n\pi}{12} = 0$ but could not solve. Most recognised $\operatorname{Im} z = 0$.

b)



$\vec{AB} = \frac{2}{3}\vec{AC}$

$\begin{pmatrix}4\\a\\2\end{pmatrix} = \frac{2}{3}\begin{pmatrix}6\\3\\b-3\end{pmatrix}$

$\therefore a = \frac{2}{3}\times 3$

$= 2$

$2 = \frac{2}{3}(b-3)$

$b-3 = 3$

$b = 6$

$\therefore a = 2, b = 6$

Lots of silly mistakes

Some solved $\begin{pmatrix}3\\a\\5\end{pmatrix} = \frac{2}{3}\begin{pmatrix}6\\3\\b-3\end{pmatrix}$ not realising $3 \neq \frac{2}{3}\times 6$

c i) $-3-4i=(x+iy)^2$

$= x^2-y^2+2xyi$

Equate Re: $-3=x^2-y^2$

Equate Im: $-4=2xy$

$-2=xy$

$\therefore x=\pm 1,\ y=\mp 2$

$\therefore 1-2i$ and $-1+2i$

Well done

ii) $$z=\frac{3\pm\sqrt{(-3)^2-4(1)(3+i)}}{2\times 1}$$

$$=\frac{3\pm\sqrt{-3-4i}}{2}$$

$$=\frac{3\pm(1-2i)}{2}$$

$= 2-i$ or $1+i$

Well done

d i) $\ddot{x}=v\dfrac{dv}{dx}$

$=-0.3x(-0.3)$

$=0.09x$

$\therefore$ motion is not simple harmonic as it is not of the form $\ddot{x}=-n^2x$

Some students could not manipulate $v=-0.3x$ to $\ddot{x}$

ii) $\dfrac{dx}{dt}=-0.3x$

$$\int_6^x \frac{1}{x}\,dx=\int_0^t -0.3\,dt$$

$$[\ln|x|]_6^x=[-0.3t]_0^t$$

$\ln x-\ln 6=-0.3t-0$

$$t=\frac{\ln x-\ln 6}{-0.3}$$

When $x=2$:

$$t=\frac{\ln 2-\ln 6}{-0.3}$$

$$=-\frac{10}{3}\ln\frac{1}{3}$$

$$=\frac{10}{3}\ln 3 \text{ seconds}$$

Some students attempted to integrate both sides to get $x=\dfrac{-0.3x^2}{2}+C$

Students who recognised how to solve the DE were generally successful

12ai) For all positive integers k, there exists some integer x such that $x^2+x-k=0$ Well done

ii) Assume for a contradiction that:

p is odd and $x^2+x-p^2=0$ has integer solutions

$\therefore p = 2k+1,\ k \in \mathbb{Z}$

Most wrote "If p is odd, then $x^2+x-p^2=0$ has integer solutions."

$x^2+x-(2k+1)^2 = x^2+x-4k^2-4k-1$

Must use "and" and no "if" to get 1mk.

Case I: x is even $\Rightarrow x = 2m,\ m \in \mathbb{Z}$

$x^2+x-p^2 = (2m)^2+(2m)-4k^2-4k-1$

$= 4m^2+2m-4k^2-4k-1$

$= 2(2m^2+m-2k^2-2k)-1$ which is odd so $\neq 0$

$\therefore x$ cannot be even

Case II: x is odd $\Rightarrow x = 2n+1,\ n \in \mathbb{Z}$

$x^2+x-p^2 = (2n+1)^2+(2n+1)-4k^2-4k-1$

$= 4n^2+4n+1+2n+1-4k^2-4k-1$

$= 2(2n^2+3n-2k^2-2k)+1$ which is odd so $\neq 0$

$\therefore x$ cannot be odd

$\therefore x$ cannot be even nor odd which contradicts $x^2+x-p^2=0$ has integer solutions

$\therefore p$ is odd $\Rightarrow x^2+x-p^2=0$ has no integer solutions

Most not successful.

Two successful attempts:

1) $\alpha+\beta=-1$ and $\alpha\beta=-p^2 \rightarrow$ odd

$\therefore \alpha, \beta$ both odd

$\therefore \alpha+\beta$ is even

2) $x(x+1) = p^2 \rightarrow$ odd

x odd $\Rightarrow x(x+1)$ even

x even $\Rightarrow x(x+1)$ even

b i) $\ell_1: \underline{r}_1(t) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\ell_2: \underline{r}_2(t) = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Equate x, y, z:

$2+t = 2+2\lambda \Rightarrow t = 2\lambda$ ①

$-t = -1-\lambda \Rightarrow t = 1+\lambda$ ②

$3+2t = 4+3\lambda$ ③

Equate ① and ②:

$2\lambda = 1+\lambda$

$\lambda = 1$

$t = 2$

Generally ok. Most recognised to equate x, y, z components

Sub into ③:

$3+2\times 2 = 7$

$4+3(1) = 7$

$\therefore \ell_1$ and ℓ_2 intersect

ii) $\underline{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\underline{b}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\cos\theta = \dfrac{\underline{b}_1 \cdot \underline{b}_2}{|\underline{b}_1||\underline{b}_2|}$

$= \dfrac{2+1+6}{\sqrt{1^2+1^2+2^2} \times \sqrt{2^2+1^2+3^2}}$

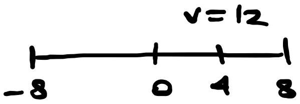
$= \dfrac{9}{\sqrt{6}\times\sqrt{14}}$

$\theta = \cos^{-1}\left(\dfrac{9}{\sqrt{6}\times\sqrt{14}}\right)$

$= 10^\circ 54'$ (n.min)

Mostly ok. Lots of arithmetic/small errors.

e)



$\ddot{x} = -n^2 x = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{1}{2}v^2 = \frac{-n^2x^2}{2} + c$

When $x = 8$, $v = 0$:

$0 = \frac{-n^2 \times 8^2}{2} + c$

$c = 32n^2$

$\therefore \frac{1}{2}v^2 = \frac{-n^2x^2}{2} + 32n^2$

When $x = 4$, $v = 12$:

$\frac{1}{2} \times 12^2 = \frac{-n^2 \times 4^2}{2} + 32n^2$

$72 = 24n^2$

$n^2 = 3$

$n = \pm\sqrt{3}$

$\therefore T = \frac{2\pi}{\sqrt{3}}$ seconds

Some considered

$x = a\sin(nt+\alpha) + c$

$\dot{x} = an\cos(nt+\alpha)$

and $\sin^2(nt+\alpha) + \cos^2(nt+\alpha) = 1$

to get

$\left(\frac{x-c}{a}\right)^2 + \left(\frac{\dot{x}}{an}\right)^2 = 1$

← more successful method

Others found the value of $nt+\alpha$ but could not use it to find n and hence T.

d i) $$\frac{1}{x(1+x^2)} = \frac{a(1+x^2)+x(bx+c)}{x(1+x^2)}$$

$$1 = a + ax^2 + bx^2 + cx$$

Equating coefficients:

$$a = 1$$

$$c = 0$$

$$a + b = 0$$

$$b = -1$$

Well done.

$\therefore$ $a = 1, b = -1, c = 0$

ii) $u = \tan^{-1} x$ $\quad v' = \frac{1}{x^2}$

$u' = \frac{1}{1+x^2}$ $\quad v = \frac{-1}{x}$

Well done.
Most can correctly apply IBP.
Some need to be careful of negatives, absolute values and +C's.

$$\int \frac{\arctan x}{x^2}\, dx = \frac{-\tan^{-1} x}{x} + \int \frac{1}{x(1+x^2)}\, dx$$

$$= \frac{-\tan^{-1} x}{x} + \int \frac{1}{x} - \frac{x}{1+x^2}\, dx$$

$$= \frac{-\tan^{-1} x}{x} + \ln|x| - \frac{1}{2}\ln|1+x^2| + C$$

13a i) $\ddot{x} = -kv^3 = v\frac{dv}{dx}$

$$-kv^2 = \frac{dv}{dx}$$

$$\int_0^x -k\,dx = \int_u^v \frac{1}{v^2}\,dv$$

$$[-kx]_0^x = \left[\frac{-1}{v}\right]_u^v$$

$$-kx + 0 = \frac{-1}{v} + \frac{1}{u}$$

$$\frac{1}{v} = \frac{1}{u} + kx$$

$$= \frac{1+kux}{u}$$

$$v = \frac{u}{1+kux}$$

Done ok.
most recognised
$\ddot{x} = v\frac{dv}{dx}$

ii) $\frac{dv}{dt} = -kv^3$

$$\int_u^v \frac{1}{v^3}\,dv = \int_0^t -k\,dt$$

$$\left[\frac{-1}{2v^2}\right]_u^v = [-kt]_0^t$$

$$\frac{-1}{2v^2} + \frac{1}{2u^2} = -kt + 0$$

$$kt = \frac{1}{2v^2} - \frac{1}{2u^2}$$

$$t = \frac{1}{2kv^2} - \frac{1}{2ku^2}$$

When $v = \frac{u}{3}$:

$$t = \frac{1}{2k(\frac{u}{3})^2} - \frac{1}{2ku^2}$$

$$= \frac{9-1}{2ku^2}$$

$$= \frac{4}{ku^2}$$

many used
$\frac{dx}{dt} = \frac{u}{1+kux}$ → generally ok

some not aware
equation needs to be
in terms of t.

b) $x = 3\sec\theta \Rightarrow \frac{x}{3} = \sec\theta \Rightarrow \cos\theta = \frac{3}{x}$

$$\frac{dx}{d\theta} = -3(\cos\theta)^{-2} \times -\sin\theta$$

$$dx = \frac{3\sin\theta}{\cos^2\theta}\, d\theta$$

$$I = \int \frac{\cancel{3}}{(\cancel{3}\cancel{\sec}\theta)^2\sqrt{(3\sec\theta)^2-9}} \times \frac{\cancel{3}\sin\theta}{\cancel{\cos^2}\theta}\, d\theta$$

$$= \int \frac{\sin\theta}{3\sqrt{\tan^2\theta}}\, d\theta$$

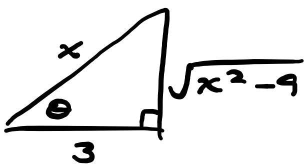
$$= \int \frac{\sin\theta}{3\tan\theta}\, d\theta$$ (valid if we pick θ to be in 1st and 3rd quad)

$$= \frac{1}{3}\int \cos\theta\, d\theta$$

$$= \frac{1}{3}\sin\theta + C$$

Some did not recognise $x = 3\sec\theta$

Those that did could integrate to $\frac{1}{3}\sin\theta + C$.



$$\therefore I = \frac{1}{3}\frac{\sqrt{x^2-9}}{x} + C$$

$$= \frac{\sqrt{x^2-9}}{3x} + C$$

Some students could not correctly draw the triangle to write back in terms of x.

c) Let $z = x + iy$:

$$\frac{z-2}{z-i} = \frac{(x-2)+iy}{x+i(y-1)} \times \frac{x-i(y-1)}{x-i(y-1)}$$

$$= \frac{x(x-2)+i(x-2)(y-1)+ixy+y(y-1)}{x^2+(y-1)^2}$$

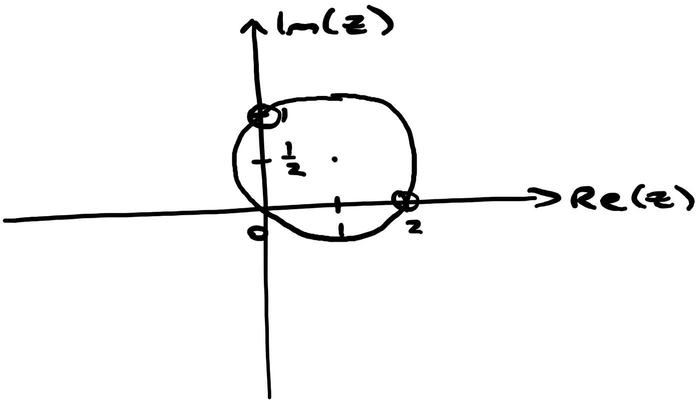
$\mathrm{Re}\left(\frac{z-2}{z-i}\right) = 0$ so

$$\frac{x(x-2)+y(y-1)}{x^2+(y-1)^2} = 0$$

$$x^2-2x+y^2-y=0$$

$$(x-1)^2+(y-\tfrac{1}{2})^2 = 1+\tfrac{1}{4}$$

$$(x-1)^2+(y-\tfrac{1}{2})^2 = \left(\tfrac{\sqrt{5}}{2}\right)^2$$



Poorly done.

Many tried to 'realise' by $\frac{z-2}{z-i} \times \frac{z+i}{z+i}$

Some thought to solve $\mathrm{Im}\left(\frac{z-2}{z-i}\right)=0$

Students who correctly identified the circle did not add open circles to exclude 2 and i

di) $\omega^5 = 1$

$$\omega^5 - 1 = 0$$

$$(\omega-1)(\omega^4+\omega^3+\omega^2+\omega+1)=0$$

Since ω is non-real, $\omega \neq 1$

$$\therefore \omega^4+\omega^3+\omega^2+\omega+1=0$$

Well done.
Students should be reminded to state that $\omega \neq 1$ as ω is non-real

ii) $(\omega-\omega^4)^2 = \omega^2 - 2\omega^5 + \omega^8$

$$= \omega^2+\omega^3-2 \quad \text{since } \omega^5 = 1$$

$$\therefore (\omega-\omega^4)^4 + 5(\omega-\omega^4)^2 + 5$$

$$= (\omega^2+\omega^3-2)^2 + 5(\omega^2+\omega^3-2)+5$$

$$= \omega^4+\omega^5-2\omega^2+\omega^5+\omega^6-2\omega^3-2\omega^2-2\omega^3+4$$
$$+5\omega^2+5\omega^3-10+5$$

$$= \omega^4+1-2\omega^2+1+\omega-2\omega^3-2\omega^2-2\omega^3+4+5\omega^2$$
$$+5\omega^3-5$$

$$= \omega^4+\omega^3+\omega^2+\omega+1$$

$$= 0$$

Students who expanded generally did well.

Some students assumed $\omega = \mathrm{cis}\frac{2\pi}{5}$ rather than any fifth root of unity.

14a) When $n=0$:

$$a_0 = \frac{0(0+1)(2\times0+1)}{2}$$

$$=0$$

$\therefore$ true for $n=0$

Many students used $n=1$ as the base case. Most could prove true for $n=k+1$, given $n=k$.

Assume true for $n=k$,

i.e. $a_k = \dfrac{k(k+1)(2k+1)}{2}$ ①

When $n=k+1$,

$a_{k+1} = a_k + 3(k+1)^2$ using recursive definition

$$= \frac{k(k+1)(2k+1)}{2} + 3(k+1)^2 \text{ using ①}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{2}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{2}$$

$$= \frac{(k+1)(2k^2+4k+3k+6)}{2}$$

$$= \frac{(k+1)(2k+3)(k+2)}{2}$$

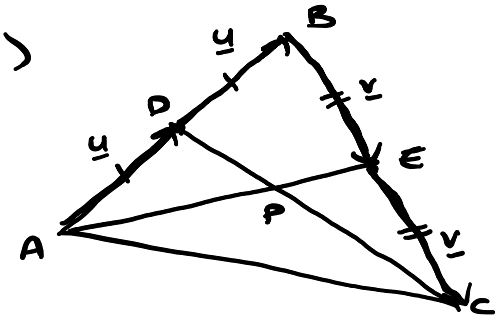
$$= \frac{(k+1)(k+1+1)[2(k+1)+1]}{2}$$

$\therefore$ true for $n=k+1$

Since it is true for $n=k+1$ whenever it is true for $n=k$, and it is true for $n=0$,

$\therefore$ by mathematical induction it is true for all integers $n \geq 0$.

b i)



$\overrightarrow{AE} = 2\underline{u} + \underline{v}$

$\therefore \overrightarrow{AP} = k(2\underline{u} + \underline{v})$

$\overrightarrow{DC} = \underline{u} + 2\underline{v}$

$\therefore \overrightarrow{DP} = \mu(\underline{u} + 2\underline{v})$

$\overrightarrow{AP} = \underline{u} + \mu(\underline{u} + 2\underline{v})$

Many students identified that $\overrightarrow{AP} = k(2\underline{u} + \underline{v})$ but did not proceed

$\therefore 2k\underline{u} + k\underline{v} = \underline{u} + \mu\underline{u} + 2\mu\underline{v}$

$(2k - 1 - \mu)\underline{u} = (2\mu - k)\underline{v}$

Since $\underline{u} \nparallel \underline{v}$, $2k - 1 - \mu = 0$ and $2\mu - k = 0$

$\therefore \mu = \frac{k}{2}$

$2k - 1 - \frac{k}{2} = 0$

$\frac{3}{2}k = 1$

$k = \frac{2}{3}$

Many students did not specify why "coefficients" of $\underline{u}$ and $\underline{v}$ can be equated

ii) $\overrightarrow{AC} = 2\underline{u} + 2\underline{v}$

$\therefore \overrightarrow{AF} = \underline{u} + \underline{v}$

$\overrightarrow{FP} = -(\underline{u} + \underline{v}) + \frac{2}{3}(2\underline{u} + \underline{v})$

$= \frac{1}{3}\underline{u} - \frac{1}{3}\underline{v}$

Generally well done

$\overrightarrow{FB} = -(\underline{u} + \underline{v}) + 2\underline{u}$

$= \underline{u} - \underline{v}$

$= 3\overrightarrow{FP}$

Since $FB \parallel FP$ and shares common point F, F, B, P are collinear

i) $z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ using De Moivre's Thm

Done well. Students should be reminded to state the use of De Moivre's Thm.

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos(n\theta)$

ii) $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$

$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

Using (i):

Generally done well.

$(2\cos\theta)^4 = 2\cos4\theta + 4\times2\cos2\theta + 6$

$16\cos^4\theta = 2\cos4\theta + 8\cos2\theta + 6$

$\cos^4\theta = \frac{1}{8}\cos4\theta + \frac{1}{2}\cos2\theta + \frac{3}{8}$

iii) $\int_0^{\pi/2} \cos^4\theta \, d\theta = \int_0^{\pi/2} \frac{1}{8}\cos4\theta + \frac{1}{2}\cos2\theta + \frac{3}{8} \, d\theta$

$= \left[\frac{\sin4\theta}{8\times4} + \frac{\sin2\theta}{2\times2} + \frac{3}{8}\theta\right]_0^{\pi/2}$

Generally well done. Students need to be careful of exact trig values.

$= \left(\frac{\sin2\pi}{32} + \frac{\sin\pi}{4} + \frac{3}{8}\times\frac{\pi}{2}\right)$

$- \left(\frac{\sin0}{32} + \frac{\sin0}{4} + \frac{3}{8}\times0\right)$

$= \frac{3\pi}{16}$

15a) When $y=0$:

$2x\ln x = 0$

$\therefore x = 0$ or $\ln x = 0$

$x = 1$

From $x=\frac{1}{e}$ to $x=1$, $y = 2x\ln x < 0$

$I = \int_{1/e}^{1} 2x\ln x \, dx$

$u = \ln x \qquad v' = 2x$

$u' = \frac{1}{x} \qquad v = x^2$

$I = \left[x^2\ln x\right]_{1/e}^{1} - \int_{1/e}^{1} \frac{x^2}{x} \, dx$

$= 1^2\ln 1 - \left(\frac{1}{e}\right)^2 \ln\left(\frac{1}{e}\right) - \left[\frac{x^2}{2}\right]_{1/e}^{1}$

$= \frac{1}{e^2} - \frac{1^2}{2} + \frac{1}{2e^2}$

$= \frac{3}{2e^2} - \frac{1}{2}$

$\therefore A_1 = \frac{1}{2} - \frac{3}{2e^2}$

From $x=1$ to $x=e$, $y = 2x\ln x > 0$

$I = \left[x^2\ln x\right]_1^e - \left[\frac{x^2}{2}\right]_1^e$

$= e^2\ln e - 1^2\ln 1 - \frac{e^2}{2} + \frac{1^2}{2}$

$= \frac{e^2}{2} + \frac{1}{2}$

$\therefore A_2 = \frac{e^2}{2} + \frac{1}{2}$

$\therefore \text{Area} = \frac{e^2}{2} - \frac{3}{2e^2} + 1$

Many forgot to check + or − areas. Otherwise, all students recognised IBP but some need to be careful with signs.

b i) When $n=1$:

$$RHS = \frac{1-(1+1)z^1+z^2}{(1-z)^2}$$

$$= \frac{(1-z)^2}{(1-z)^2}$$

$$= 1$$

$$= LHS$$

$\therefore$ true for $n=1$

Common mistake: wrong base case

Many stopped at inductive step after getting a common denominator

Assume true for $n=k$,

i.e. $1+2z+3z^2+\ldots+kz^{k-1} = \dfrac{1-(k+1)z^k+kz^{k+1}}{(1-z)^2}$ ①

When $n=k+1$,

$$LHS = 1+2z+3z^2+\ldots+kz^{k-1}+(k+1)z^k$$

$$= \frac{1-(k+1)z^k+kz^{k+1}}{(1-z)^2} + (k+1)z^k \quad \text{using ①}$$

$$= \frac{1-(k+1)z^k+kz^{k+1}+(k+1)z^k(1-2z+z^2)}{(1-z)^2}$$

$$= \frac{1-\cancel{(k+1)}z^k+kz^{k+1}+\cancel{(k+1)}z^k-2(k+1)z^{k+1}+(k+1)z^{k+2}}{(1-z)^2}$$

$$= \frac{1-(k+2)z^{k+1}+(k+1)z^{k+2}}{(1-z)^2}$$

$$= RHS$$

$\therefore$ true for $n=k+1$

Since true for $n=k+1$ whenever it is true for $n=k$, and it is true for $n=1$, $\therefore$ by mathematical induction it is true for all integers $n \geq 1$.

ii) Sub $z = \frac{1}{2}$:

$$1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)^3 + \dots + n\left(\tfrac{1}{2}\right)^{n-1} = \frac{1-(n+1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}}{\left(1-\frac{1}{2}\right)^2}$$

$$\frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{n}{2^{n-1}} = \frac{1 - \frac{n+1}{2^n} + \frac{n}{2^{n+1}}}{\frac{1}{2^2}} - 1$$

$$2 + \frac{3}{2} + \frac{4}{2^2} + \dots + \frac{n}{2^{n-2}} = \frac{2 - \frac{n+1}{2^{n-1}} + \frac{n}{2^n}}{\frac{1}{2^2}} - 2$$

Q ii onwards not attempted by all

$$= \frac{2(2^n) - 2(n+1) + n}{\frac{1}{2^2} \times 2^n} - 2$$

most attempts identified $z = \frac{1}{2}$, but struggled to manipulate to get the result

$$= \frac{2^{n+1} - n - 2}{2^{n-2}} - 2$$

$$= 2^3 - \frac{n+2}{2^{n-2}} - 2$$

$$= 6 - \frac{n+2}{2^{n-2}}$$

iii) $$1 + 2z + 3z^2 + \dots + nz^{n-1} = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} \quad \text{from (i)}$$

$$= \frac{\frac{1}{z} - (n+1)\frac{z^n}{z} + \frac{nz^{n+1}}{z}}{\frac{1}{z} - \frac{2z}{z} + \frac{z^2}{z}}$$

most who attempted knew to divide (i) by z - students should be reminded to be more explicit with show questions

$$= \frac{z^{-1} - (n+1)z^{n-1} + nz^n}{z^{-1} - 2 + z}$$

iv) $1+2\,\text{cis}\,\theta+3(\text{cis}\,\theta)^2+\ldots+n(\text{cis}\,\theta)^{n-1}=\dfrac{(\text{cis}\,\theta)^{-1}-(n+1)(\text{cis}\,\theta)^{n-1}+n(\text{cis}\,\theta)^n}{(\text{cis}\,\theta)^{-1}-2+(\text{cis}\,\theta)}$

Using De Moivre's thm:

$$1+2\,\text{cis}\,\theta+3\,\text{cis}\,2\theta+\ldots+n\,\text{cis}(n-1)\theta=\frac{\text{cis}(-\theta)-(n+1)\text{cis}(n-1)\theta+n\,\text{cis}(n\theta)}{\text{cis}(-\theta)-2+\text{cis}\,\theta}$$

$$=\frac{\text{cis}(-\theta)-(n+1)\text{cis}(n-1)\theta+n\,\text{cis}(n\theta)}{2\cos\theta-2}$$

Equate real parts:

$$1+2\cos\theta+3\cos2\theta+\ldots+n\cos(n-1)\theta=\frac{\cos(-\theta)-(n+1)\cos(n-1)\theta+n\cos(n\theta)}{2\cos\theta-2}$$

$$\sum_{k=1}^{n}k\cos(k-1)\theta=\frac{\cos\theta-(n+1)\cos(n-1)\theta+n\cos(n\theta)}{-2(1-\cos\theta)}$$

$$=\frac{(n+1)\cos(n-1)\theta-n\cos(n\theta)-\cos\theta}{2(1-\cos\theta)}$$

Most attempts recognised taking the real parts of both sides. Students need to be careful of arithmetic errors.

16a i) $(\sqrt{a}-\sqrt{b})^2 \geq 0$

$a - 2\sqrt{ab} + b \geq 0$

$a + b \geq 2\sqrt{ab}$

well done

ii) Let $a = \tan^2 x$, $b = 1$:

$\tan^2 x + 1 \geq 2\sqrt{\tan^2 x \times 1}$

$\sec^2 x \geq 2|\tan x|$

$\geq 2\tan x$

most students recognised $a = \tan^2 x$, $b = 1$

Students should be reminded that $\sqrt{\tan^2 x} = |\tan x| \geq \tan x$

iii) Let $a = \cot^2 x$, $b = 1$:

$\cot^2 x + 1 \geq 2\sqrt{\cot^2 x \times 1}$

$\operatorname{cosec}^2 x \geq 2|\cot x|$

$\geq 2\cot x$

many attempted induction – 1 mk for correct base case.

$\therefore (\sec^2 x)^n \geq (2\tan x)^n$ and $(\operatorname{cosec}^2 x)^n \geq (2\cot x)^n$ since $x \in (0, \frac{\pi}{2})$

$\therefore \sec^{2n} x \geq 2^n \tan^n x$ and $\operatorname{cosec}^{2n} x \geq 2^n \cot^n x$

$\sec^{2n} x + \operatorname{cosec}^{2n} x \geq 2^n \tan^n x + 2^n \cot^n x$

$\geq 2^n(\tan^n x + \cot^n x)$

$\geq 2^n \times 2\sqrt{\tan^n x \cot^n x}$ using (i)

$\geq 2^n \times 2\sqrt{1}$

$\geq 2^{n+1}$

Two clever attempts:

1) $\sec^{2n} x \geq 2^n \tan^n x$ using (ii)

$\frac{\sec^{2n} x}{\tan^n x} \geq 2^n$, $x \neq 0$

$\frac{2}{\cos^n x \sin^n x} \geq 2^{n+1}$ ①

$\sec^{2n} x + \operatorname{cosec}^{2n} x \geq 2\sqrt{\sec^{2n} x \operatorname{cosec}^{2n} x}$ using (i)

$\geq \frac{2}{\cos^n x \sin^n x}$

$\geq 2^{n+1}$ using ①

2) $\sec^{2n} x + \operatorname{cosec}^{2n} x \geq 2\sqrt{\sec^{2n} x \operatorname{cosec}^{2n} x}$ using (i)

$\geq \frac{2}{\cos^n x \sin^n x}$

$\geq \frac{2^{n+1}}{\sin^n 2x}$

$\geq 2^{n+1}$ since $\sin^n 2x \leq 1$ as $\sin 2x \leq 1$

b i) $u = x^{n-1}$ $\quad v' = x\sqrt{a^2-x^2}$

$u' = (n-1)x^{n-2}$ $\quad v = \frac{-1}{2}\frac{(a^2-x^2)^{3/2}}{3/2}$

$$= \frac{-1}{3}(a^2-x^2)^{3/2}$$

$$I_n = \left[\frac{-x^{n-1}(a^2-x^2)^{3/2}}{3}\right]_0^a + \int_0^a \frac{(n-1)x^{n-2}}{3}(a^2-x^2)^{3/2}\,dx$$

$$= \frac{-a^{n-1}(a^2-a^2)}{3} + \frac{0^{n-1}(a^2-0^2)}{3} + \frac{(n-1)}{3}\int_0^a x^{n-2}(a^2-x^2)^{3/2}\,dx$$

$$= \frac{n-1}{3}\int_0^a x^{n-2}\sqrt{a^2-x^2}\,(a^2-x^2)\,dx$$

$$= \frac{n-1}{3}\int_0^a a^2x^{n-2}\sqrt{a^2-x^2} - x^n\sqrt{a^2-x^2}\,dx$$

$$= \frac{n-1}{3}\left[a^2 I_{n-2} - I_n\right]$$

Some did not attempt Q6

$$I_n + \frac{n-1}{3}I_n = \frac{a^2(n-1)}{3}I_{n-2}$$

Those who recognised the correct u and v' had more success, though many wrote v incorrectly

$$3I_n + nI_n - I_n = a^2(n-1)I_{n-2}$$

$$(n+2)I_n = a^2(n-1)I_{n-2}$$

$$I_n = \frac{a^2(n-1)}{n+2}I_{n-2}$$

ii) $$I_{2n} = \frac{a^2(2n-1)}{2n+2}I_{2n-2}$$

Only one attempt on the right track. many tried to find I_0 the hard way.

$$= \frac{a^2(2n-1)}{2n+2} \times \frac{a^2(2n-2-1)}{2n-2+2}I_{2n-4}$$

$$= \frac{a^2(2n-1)\times a^2(2n-3)}{(2n+2)\times 2n} \times \frac{a^2(2n-4-1)}{2n-4+2} \times \dots \times \frac{a^2(2-1)}{2+2}I_0$$

$$= \frac{(a^2)^n(2n-1)(2n-3)(2n-5)\dots 1}{(2n+2)2n(2n-2)\dots 4}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$= \frac{(a^2)^n(2n-1)(2n-3)(2n-5)\dots 1}{(2n+2)2n(2n-2)\dots 4} \times \frac{1}{4}\pi \times a^2$$

$$= \frac{a^{2n+2}\pi}{4} \times \frac{(2n-1)(2n-3)(2n-5)\dots 1}{2^n(n+1)n(n-1)\dots 2}$$

$$= \frac{a^{2n+2}\pi}{2^{n+2}} \times \frac{2n(2n-1)(2n-2)(2n-3)\dots 2\times 1}{(n+1)!\times 2n(2n-2)(2n-4)\dots 2}$$

$$= \frac{a^{2n+2}\pi}{2^{n+2}} \times \frac{(2n)!}{(n+1)! \times 2^n \times n(n-1)(n-2)\cdots 1}$$

$$= \frac{a^{2n+2}\pi\,(2n)!}{2^{n+2} \times 2^n (n+1)!\,n!}$$

$$= \pi \left(\frac{a}{2}\right)^{2n+2} \frac{(2n)!}{n!\,(n+1)!}$$

iii) $c_n = \frac{1}{n+1}\binom{2n}{n}$

$$= \frac{1}{n+1} \frac{(2n)!}{n!(2n-n)!}$$

$$= \frac{(2n)!}{n!\,(n+1)!}$$

$$= \frac{1}{\pi} \times \pi \left(\frac{2}{2}\right)^{2n+2} \frac{(2n)!}{n!\,(n+1)!}$$

$$= \frac{1}{\pi} \int_0^2 x^{2n}\sqrt{2^2 - x^2}\,dx$$ using (ii) and $a = 2$

$$= \frac{1}{\pi} \int_0^2 x^{2n}\sqrt{4 - x^2}\,dx$$

Only two attempts but both successful. Q iii is easier than Q ii but more students attempted Q ii than Q iii